Hints and Answers to Selected Exercises

Note to students: The selected answers below give you some indication about whether you are on the right track. They are NOT intended to be the type of answers you would turn in to an instructor. Your complete answers should contain all the work towards obtaining an answer, as described in the Message to Students. Some exemplary answers are included and they are noted as such.

1.1 More About Ancient Numeration Systems

5. a. 2113 b. 185 c. 1507

   d. MMLXVI e. LXXVIII f. DCXXV

6. a. 1994 b. 49 c. 409

1.2 Place Value

1. a. 30; 5 b. 35 c. 35 d. 43 e. 436

   f. 456 g. 4566 h. 45 665 i. 234 j. 23,470

2. No; The <u>ones</u> place is the place value which gives a sort of symmetry to all the place values.

3. a. True b. It depends on the placement of the decimal point.

1.3 Bases Other Than Ten

2. a. 22_{four} b. 20_{five} c. 12_{eight}

3. a. 10_{four} b. 10_{eight} c. 10_{twenty} d. 10_b

   e. 100_b f. 1100_b g. 1002_{three} h. 430_{five}

   i. 1000101_{two} j. 1000_{twelve}

4. 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011, 10100

5. a. There cannot be a “4” with base three.

   b. There cannot be a “7” with base seven.

   c. There cannot be a “5” with base four.

6. a. forty-three b. incorrect symbol (no “7” in base four)

c. two hundred seven and twenty-four ten-thousandths

d. eight e. sixty-four f. twenty-five and two-thirds

7. Base two will usually have a numeral with more digits; the symbols for zero and one are the exceptions.

8. Base six and greater

10. Base four. ..., fin na, fin obi, fin fin, fin mus, mus na, mus obi, mus fin, mus mus, obi na na, obi na obi, obi na fin, obi na mus, obi obi na,...

11. a. twenty b. twenty c. twelve d. sixty

1.4 Multibase Blocks

2. You should have 2 flats, 3 longs, and 4 small cubes.

1.5 Operations in Different Bases

3. b. 611_{nine} d. 176_{eleven}

4. b. 506_{seven} d. 11_{eleven}

5. b. 240_{six} d. 12_{four}

6. Because we don't know our basic multiplication and division facts in bases other than ten.

8. a. $2\frac{7}{8}$ b. $9\frac{11}{16}$ c. $40\frac{3}{12} = 40\frac{1}{4}$

9. a. 0.3_{twelve} b. 0.9_{twelve} c. 0.11_{two} d. 0.2_{eight}

1.7 Teaching Place Value for Numbers Less Than 1

1. 73.5 is 73.5 <u>ones</u>, is 7.35 tens, is .735 hundreds, is .0735 thousands; it is also 735 tenths, or 7350 hundredths. (Think of this as: $73.5 worth 7.35 ten dollar bills, .735 hundred dollar bills, .0735 thousand dollar bills, <u>and</u> 735 dimes, 7350 pennies.)

3. a. 8.3_{nine} c. 222030.02

2.2 Finding Sums, Remainders, and Differences

2. Student 6 (Case C): (b) The student is using a variation of adding on to the second number to reach the first number. The student adds an 8 to get to 15, then adds tens to get to 65. Another example of this method might be 43 - 9: Add 4 to get 13, then 10 more to 23, 10 more

to 33, ten more to 43, so I've added 3 tens and a 4; 34. (This is sometimes called shopkeepers math, because if you gave a shopkeeper $10 for something that cost $6.25, she might count out money saying, “$6.25, $7.00, $8.00, $9.00, $10.00” while giving you 3 quarters and 3 one dollar bills.) (c) This is not too difficult to remember, but it might, for some people, be easier to add to ten, then tens, then whatever more is need: 43 - 9 would be 1 to get to 10, then 30 to get to 40, then 3 more: 1 + 30 + 3 is 34.

2.5 Children Find Products and Quotients

1. The first student's method is related to estimating because she first estimated the quotient to be between 2080 and 2240, then found the remainder to obtain the exact answer.

4. The method used here is taught by some teachers, because the steps are so much more easily understood than the traditional algorithm most of us learned. The major advantage of this method is that it can be easily understood, based on the subtraction notion of division. We ask: How many 27's are in 3247? We know there are at least 100 because 27 x 100 = 2700 is less than 3247, so we first subtract 100 27's. We could next take away 10 27's, or we could take away 20 27's; the choice depends on one's number facility. This is another advantage. The disadvantage is that it takes longer than the standard algorithm (but not if one counts all the additional time needed to learn the standard algorithm for division).

5. This procedure was invented by a very precocious child. Most first graders would not have been able to do this. However, prospective teachers should recognize that they will sometimes have very bright students whose thinking is not always easily followed.

2.6 Teaching Notes: Developing Number Sense

1. a. 135 + 98, since each addend is greater than the corresponding addend in 114 + 92

   d. 0.0016 + 0.313, since the 3 tenths in the sum is itself greater than 0.0358

2. Less, more. You give the explanation.

3. If 32 is rounded to 30, and the 83 is not rounded, the answer is off by 2 x 83 or about 160. If 83 is rounded to 80 and the 32 is not rounded, the answer is off by 3 x 32 or about 90, so rounding the 83 yields a closer estimate. (The temptation is to round the 32 because "only 2 is being dropped, but if 83 is rounded to 80, 3 is being dropped.")

6. a. 47 and 52 are "compatible," adding about to 100, and 36 and 69 are almost compatible, with sum about 100. Sum about 200 + 20 or 30: 220 or 230.

   b. About 40 x 40, or 1600 d. About 35,000 ÷ 50, or 700

8. 0.68 x 5 is trivial compared to 0.34 x 150, which is about a third of 150.

3.1 Operating on Whole Numbers and Decimals, First Set

1. d. 2912 ÷ 8

Here is one way this could be acted and written, using the sharing notion of division. The small block is used to represent 1.

Place or draw 2 blocks, 9 flats, 1 long, and 2 small blocks. Write:

$$8\overline{)2912}$$

Think of how to place the blocks in 8 piles with the same amount in each pile. Change the 2 blocks to 20 flats; there are now 29 flats; place 3 in each of eight piles. 3

$$\begin{array}{r} 8\overline{)2912} \\ \underline{24} \\ 5 \end{array}$$

There are 5 remaining flats. Change them to longs; there are now 51 longs.

Distribute the longs; there will be 6 longs in each pile, with 3 longs remaining.

$$\begin{array}{r} 36 \\ 8\overline{)2912} \\ \underline{24} \\ 51 \\ \underline{48} \\ 3 \end{array}$$

Change the 3 longs to 30 units. There are now 32 units. Distribute them to the piles, 4 per pile.

```
   364
8)2912
  24
   51
   48
    32
    32
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In each pile there are 3 flats, 6 longs, and 4 units per pile. The quotient is therefore 364.

4.1 Number Neighbors

2. $\frac{7}{12}$ is more than $\frac{1}{2}$ and $\frac{5}{8}$ is more than $\frac{1}{2}$, so the sum is more than 1. But $\frac{23}{24}$ is less than 1, so it is incorrect.

8. a. $\frac{2}{5}$ is less than $\frac{1}{2}$ and $\frac{3}{5}$ is greater than $\frac{1}{2}$ so $\frac{1}{2}$ is in between.

10. a. 0.125 = 12.5% k. 0.66666.... = $66\frac{2}{3}$ %

11. a. Slightly less than $\frac{1}{4}$ d. slightly more than $\frac{1}{3}$

12. d. This is about $\frac{2}{3}$ of 24 which is 16.

4.2 Mental Computation and Estimation

7. First row: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{3}{4}$; second row: $\frac{2}{3}$ $\frac{1}{8}$ $\frac{1}{5}$ $\frac{1}{10}$

8. 20% $33\frac{1}{3}$% 12.5% $66\frac{2}{3}$%

9. a. $\frac{1}{2}$; 60%c. 0; 0.6% = .006

10. Estimates should be in the neighborhoods of these percents: A) 25%; B) 11%; C) 18%; D) 10%; E) 36%

11. a. The price is about $\frac{1}{4}$ off \$50, or \$12.50 off. About \$37.50.

5.1 Notes for Teaching: Meanings of $\frac{a}{b}$

1. Infinitely many ways!

2. d. One possibility: A region could be marked into 4 equal pieces, with none shaded.

   e. 3 rectangular regions are shaded (showing that 3 = $\frac{3}{1}$)

3. Be sure to contrast parts a and b. In part a, the unit is the whole piece of licorice whip. In part b, the unit is the segment between any two consecutive whole numbers, and the labelling convention would have the $\frac{3}{4}$ in only one place (the "other" three-fourths of pieces would be labelled $1\frac{3}{4}$, $2\frac{3}{4}$, $3\frac{3}{4}$, etc.

5. a. Two square regions, each marked into 2 equal pieces, with 3 of the pieces shaded.

   b. Start with 3 square regions; how you show the division can vary. Think of how 3 brownies could be shared by 2 people. One way would be to cut one brownie in half: $3 \div 2$. Each person gets $\frac{3}{2}$, or $1\frac{1}{2}$ brownies

6. c. Hexagon = $\frac{1}{2}$; trapezoid = $\frac{1}{4}$; blue rhombus = $\frac{1}{6}$; triangle = $\frac{1}{12}$

11. Yes, it is $\frac{1}{4}$. Standing alone, one would be distracted by the unequal pieces, but since it is easily compared to the first rectangular region, one can tell that the shaded region in both cases is $\frac{1}{4}$.

12. Part b could be an interesting mini-project if your course calls for such.

13. The key is realizing that the unit for the fraction is some unspecified set of asterisks—what is here is only $\frac{5}{6}$ of the unit set. If the asterisks are divided into 5 sets. . . .

16. a. Begin by partitioning the rectangle into 8 equal-sized parts. Why 8?

5.2 Equivalent (Equal) Fractions

4. $\frac{6}{10}$

5. a. $\frac{5}{8}$ b. $\frac{45}{32}$ c. $\frac{25}{16}$ d. $\frac{xz^3}{y}$ e. 1.3×10^{11}

5.3 Relating Fractions and Decimals

1. a. 0.375 b. 2.3 c. 0.428571428571428571... $= .\overline{428571}$

   d. $0.27272727... = .\overline{27}$

2. a. $\frac{5}{8}$ b. $\frac{49}{100}$ c. $\frac{274}{3}$

   d. $\frac{17}{10}$ e. $\frac{1}{1}$ f. $\frac{28}{30}$ or $\frac{14}{15}$

3. a. Yes, infinitely many—can you find ten? How do you know $\frac{1}{2}$

works?

b. Yes, infinitely many—can you find ten?

4. a. 0.45

9. c. Sam, Chris, Lars

6.1 Adding and Subtracting Fractions

6. a. Hint: The fractions add to $\frac{16}{24}$, or $\frac{8}{12}$.

7. One solution would have the first 4 pizzas cut into fourths, and the last one into eighths.

10. a. $\frac{3}{4}+\frac{1}{4}$ = 1, and 1 + $\frac{5}{6}$ = $1\frac{5}{6}$

6.3 Notes for Teaching: Multiplication Does Not Always Make Bigger

6. a. $\frac{1}{6}$ x 12 = 2   b. 12 x $\frac{1}{6}$ = 2

7. For 4 x $\frac{2}{3}$, you should show 4 sets of 2 blue rhombuses (or 2 squares if you are not using Pattern Blocks), but for $\frac{2}{3}$ x 4, you should start with 4 hexagons (or squares) and take $\frac{2}{3}$ of that. (In each case, of course, you end up with $2\frac{2}{3}$.)

10. Each "undoes" the other, when multiplying. That is, their product gives 1, which is the identity for multiplication.

13. a. A recipe calls for $2\frac{3}{4}$ cups of sugar. Anh is making only $\frac{1}{4}$ of the recipe. How much sugar will she use?

6.4 Division by a Fraction

7. To find $\frac{1}{2} \div \frac{3}{4}$ ask: How many $\frac{3}{4}$s are in $\frac{1}{2}$? Your drawing can be used to show that $\frac{2}{3}$ of one $\frac{3}{4}$ is in $\frac{1}{2}$. Note that while $\frac{1}{2}$ and $\frac{3}{4}$ each refer to a part of the unit, the $\frac{2}{3}$ refers to the amount of $\frac{3}{4}$ that is shaded. It does NOT refer back to the unit, or square region in this case. That is, the $\frac{2}{3}$ that is showing is a part of the $\frac{3}{4}$ that is showing.

13. a. The $\frac{2}{3}$ refers to the $\frac{3}{4}$. The $\frac{3}{4}$ refers to the unit. The $\frac{1}{2}$ refers to the unit.

    b. The $\frac{1}{2}$ refers to the unit. The $\frac{3}{4}$ refers to the unit. The $\frac{2}{3}$ refers to the $\frac{3}{4}$.

(Note that the answers to a and b are the same.)

7.1 The Set of Integers

1. a. The opposite of (the opposite of 6) is the opposite of ($^{-}6$) is 6.

   c. Yes, if a is negative. For example, if a = $^{-}3$, then $^{-}$a = $^{-}$($^{-}3$) is 3.

2. a. $^{-}10$ c. 0   e. $^{-}10$

3. a. $\frac{1}{3}$ d. $\frac{1}{9}$ h. 11.5

7.3 Multiplying and Dividing Integers

3. d. 11   e. $^{-}\frac{2}{3}$ h. 1

7.4 The Rational Number System as a Field

2. $^{-}(^{-}\frac{3}{5})$ or $\frac{3}{5}$; 0; $^{-}14$

3. $-\frac{5}{3}$; $\frac{1}{14}$

4. No, the set of odd numbers is not a field, because the odd numbers are not closed under addition (and other properties also fail).

   No, because the even numbers do not have a multiplicative identity (nor do integers have multiplicative inverses.

   No, integers do not have multiplicative inverses.

6. Test for Property 1: Is addition commutative? Chose two elements, say # and &.

   # + & = @, according to the table. & + # = @ also. So # + & = & + #. (Try other pairs also.)

8.1 Factors and Multiples

1. c. v = tx for some whole number x.

2. a. 216x = 2376

   b. Solve 144x = 3456, which involves seeing whether 3456 ÷ 144 (or $\frac{3456}{144}$) is a whole number.

8.2 Prime and Composite Numbers

4. 3 is a factor of 15, so 15 has at least these three factors: 1, 15, and 3. Hence it is not prime.

5. 1 and 829 are factors of 829.

9. (for n = 8) Since 8 = 8 x 1 or 2 x 4 or 1 x 8 or 4 x 2, the chairs can be arranged in 4 ways.

10. The reason we know that none of 2, 3, 5, 7, 11, 13, and 17 is a factor of the number $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) + 1$ is that this argument can be used for each number (letting x stand for any one of the numbers in this list):

x is a factor of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, and x is not a factor of 1. By exercise 2e in the previous section, x cannot be a factor of the sum of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ and 1.

8.3 Factorization Into Primes

2. a. $5850 = 2 \cdot 3^2 \cdot 5^2 \cdot 13$ b. $256 = 2^8$ c. $2835 = 3^4 \cdot 5 \cdot 7$

   d. $10^4 = 2^4 \cdot 5^4$ e. $17280 = 2^7 \cdot 3^3 \cdot 5$

   f. A factor tree shows all the prime factors, but most often it does not show all the possible composite factors of the number.

3. Think about 11^m. This number, in prime factorization, has only 11 and powers of 11 as factors. Now think about 13^n. This number, in prime factorization, has only 13 and powers of 13 as factors. The fundamental theorem of arithmetic says that prime factorizations are unique. 13 does not appear in the list of factors of 11^m. So 13 cannot be a factor of 11^m for any value of m. This says that 13^n cannot equal 11^m for any values of m and n.

6. a. $2^8 \cdot 7$ could not be a factor of m, since m does not have 7 in its prime factorization (which is unique, and hence could not have another factorization with a 7 in it).

7. 5 and 7 (since 35 = 5x7).

8. a. 6 b. 12 c. 60 (get the prime factorization of n)

   d. 288 e. 49 f. 7

   g. 91 (Why are the answers to e, f, and g so different?)

9. a. 11 b. m + 1 e. (m+1)(n+1)(s+1)

10. For a number to have 60 factors, the product of each exponent plus 1 from the prime factorization must be 60. $11^1 \cdot 13^{29}$ is one number that has 60 factors because (1 + 1)(29 + 1) = 60, $17^4 \cdot 31^{11}$ is another number that has 60 factors. $2^2 \cdot 3^4 \cdot 5^3$ is a third number.

11. Similar to Problem 10, but the number must have 11^2 as a factor.

    $121 = 11^2$. We need a number of the form p^2q^7 where p and q are primes. So multiply 11^2 by the seventh power of a prime.

8.4 Finding Out Whether a Number is Prime

1. a. 2, 3, 4, 6, 8, 9 are (some of the) factors of 43056

   b. 2, 3, 6, 9 are (some of the) factors of 700010154

3. For 3, there are three possibilities for a. One of them is 1.

4. a. Try 18. 2 and 6 are factors of 18, but 12 is not.

   b. Try 24. 3 and 6 are factors of 24, but 18 is not.

6. a. 24 is a factor, because 3 and 8 are factors and 3 and 8 are relatively prime.

   b. 24 is not a factor, because...

   c. 18 is not a factor, because... d. 18 is a factor, because...

11. No, because...

12. None is a prime.

13. Of every two consecutive whole numbers, one will be odd and the other...., so....

14. For example, 4 = 2 + 2; 6 = 3 + 3; 8 = 3 + 5; 10 = 3 + 7 or 5 + 5;....

8.5 Greatest Common Factor; Least Common Multiple

1. a. For example, $24 \cdot 2^3 \cdot 3$, or $7 \cdot 2^3 \cdot 3$, or $59^2 \cdot 2^3 \cdot 3$

2. a. $5^2 \cdot 7^3 \cdot 13^2$ and 5 b. $37^6 \cdot 47^5 \cdot 67^6 \cdot 71$ and $37^4 \cdot 47^5$

6. E.g., m = 2, n = 3; m = 4, n = 6; m = 6, n = 9;...

7. Hint: Think of “greatest common factor” in this order: First “factor,” then “common factor,” and finally “greatest common factor.”

8. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{21}{25}$

d. $\frac{9}{8}$ e. $\frac{3^2}{2}$ f. $\frac{y^2}{x}$

10. a. $\frac{65}{72}$ b. $\frac{3}{8}$ c. $\frac{83}{60}$

d. $\frac{x^3 + 2xy}{x^2y^2}$, which can be simplified e. $\frac{33.3\times10^5}{6\times10^3}$, or 5.55 x 10^2

13. c. $\frac{49}{50}$, $\frac{35}{36}$; $\frac{37}{42}$.

d. Yes. For $\frac{81}{86}$, a common factor of 81 and 86 would have to be a common factor of 5–that is, a common factor would have to be 1 or 5. 5 is not a common factor, so there are no common factors of 81 and 86 besides 1 and so they are relatively prime.

15. 150 and 151 are one example; give another.

16. E.g., 91 (do not use it as your answer).

17. a. 115